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# Linear dynamic model for porous media saturated by two immiscible fluids

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**Abstract**

A linear isothermal dynamic model for a porous medium saturated by two immiscible fluids is developed in the paper. In contrast to the mixture theory, phase separation is avoided by introducing one energy for the porous medium. It is an important advantage of the model based on one energy approach that it can account for the couplings between the phases. The volume fraction of each phase is characterized by the porosity of the porous medium and the saturation of the wetting phase. The mass and momentum balance equations are constructed according to the generalized mixture theory. Constitutive relations for the stress, pore pressure are derived from the free energy function. A capillary pressure relaxation model characterizing one attenuation mechanism of the two-fluid saturated porous medium is introduced under the constraint of the entropy inequality. In order to describe the momentum interaction between the fluids and the solid, a frequency independent drag force model is introduced. The details of parameter estimation are discussed in the paper. It is demonstrated that all the material parameters in our model can be calculated by the phenomenological parameters, which are measurable. The equations of motion in the frequency domain are obtained in terms of the Fourier transformation. In terms of the equations of motion in the frequency domain, the wave velocities and the attenuations for three P waves and one S wave are calculated. The influences of the capillary pressure relaxation coefficient and the saturation of the wetting phase on the velocities and attenuation coefficients for the four wave modes are discussed in the numerical examples.


*Keywords:* Porous media; Immiscible fluids; Dynamic model; Capillary pressure; Entropy inequality

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## 1. Introduction

The dynamic response of porous media is of interest in various areas such as geophysics, civil engineering, ocean engineering, petroleum engineering and environmental engineering. It is pointed out in a historical review (de Boer, 1996) of the subject that two kinds of theories have been developed and used up to now, namely, Biot's theory (Biot, 1956a,b, 1962) and the mixture theory (Morland, 1972; Bedford and Drumheller, 1978; Bowen, 1980, 1982; Passman et al., 1984). The important difference between Biot's theory and the mixture theory is the coupling between state variables of the solid and the fluid phases. Bowen (1982) showed that if the coupling parameter $Q$ introduced by Biot (1956a) is neglected, then, the mixture theory is equivalent to Biot's theory. A comparative study by Schanz and Diebels (2003) shows good agreement between Biot's theory and the mixture theory in the case of incompressible constituents, while for the case of compressible constituents there is significant discrepancy between the two theories. Clearly, this is due to the fact that in the incompressible case the constitutive coupling terms in Biot's theory vanish and thus, the two approaches are equivalent.

Based on the work of von Terzaghi (1923), Biot (1956a,b, 1962) presented a theoretical description of a porous medium saturated by one fluid. In deriving the equations of motion for the porous medium, Biot introduced the Lagrangian viewpoint and the concept of generalized coordinates. Biot derived the constitutive relation of the porous medium from a single free energy. Biot also extended his theory to the anisotropic medium (Biot, 1955), poro-viscoelastic medium (Biot, 1956c) and non-linear poroelastic medium (Biot, 1972). Besides, Biot's theory was extended to the unsaturated porous medium by Brutsaert (1964), Berryman et al. (1988) and Santos et al. (1990), Hanyga (2004), Hanyga and Lu (2004). As mentioned above, a striking characteristic of Biot's theory is the constitutive coupling between the solid skeleton and the pore fluid, which can account for both mechanical interactions between bulk components and the complex interface phenomena in the porous medium. Furthermore, in contrast to some recent model (Hassanizadeh and Gray, 1990), since the coupling parameter in Biot's model can be determined by appropriate ideal experiments (Biot and Willis, 1957), the phase interaction effects can be taken into account without an explicit reference to the interfaces. However, since Biot introduced his theory in the frame of the Lagrangian mechanics, the restrictions on the dissipative mechanisms such as drag force and capillary pressure hysteresis have to be imposed by intuitive consideration.

The generalized mixture theory can also be used to describe the behavior of porous media saturated by fluids. An important feature of the generalized mixture theory is the introduction of the concept of volume fraction to characterize the microstructure of the porous medium (Morland, 1972). According to the mixture theory, at the macro-scale level, each phase can occupy the same point simultaneously in an amount determined by its volume fraction. Each phase is also characterized locally by two independent densities: a true density with respect to the true volume occupied by the phase and an bulk density with respect to the total volume occupied by the porous medium. Another important assumption of the generalized mixture theory is phase separation, which has the following implications: (1) the total energy and entropy are a sum of all the partial energies and entropies associated with the phases; (2) the partial energy and entropy of each phase depend only on the state of that phase. In effect, phase separation is only reasonable for some special cases such as particulate media. The assumption of phase separation is too restrictive in the context of general porous media since it ignores the interface energy generated by the coupling between the fluids and the solid. To circumvent the limitation of phase separation in the mixture theory, Hassanizadeh and Gray (1990), Muraleetharan and Wei (1999), Wei and Muraleetharan (2002) treated the interfaces and the common contact lines as separate phases endowed with mass, density and energy. In this way, the interaction effect between each phase can be taken into account. However, this approach introduces additional thermodynamical variables such as volume density of interface area, which is beyond the reach of the current experiments.

In order to overcome the drawbacks of both Biot's theory and the generalized mixture theory, Hanyga (2004) recently put forward a general dynamic model for porous media saturated by two immiscible fluids. Central to his theory is the idea of using one energy and one entropy for the porous medium as well as borrowing the concept of volume fraction from the mixture theory. The entropy inequality and the Coleman–Noll method (Coleman and Noll, 1963) were utilized to derive the constitutive relations for the porous medium. Also, the possible models for capillary pressure hysteresis and drag force constrained by the entropy inequality were discussed. The focus of this contribution is to derive an isothermal dynamic model for the porous medium saturated by two immiscible fluids. Following Biot (1956, 1962) and Hanyga (2004), the single energy assumption is adopted here to avoid the phase separation assumption. The volume fraction of each phase is characterized by the porosity of the porous medium and the saturation of the wetting phase. As in Biot's theory, the porosity of the porous medium is assumed to be a material constant and does not undergo variation during the dynamic process. The momentum and mass balance equations here are assumed the same forms as in the mixture theory. The constitutive relations for the stress, pore pressures are derived from the entropy inequality. In terms of the entropy inequality, a capillary pressure relaxation model is introduced as one attenuation mechanism for the porous medium. According to this model, in equilibrium, the increment of the capillary pressure is determined by the saturation increment, while in a dynamic process the increment of the capillary pressure depends additionally on the time derivative of the saturation. In order to describe the interaction between the solid and the fluids, a frequency independent drag force model is introduced in the frame of entropy inequality. Parameter estimation is also discussed in the paper. In order to determine the parameters involved in our model, some new experiments are introduced. All the material parameters in our model can be calculated by the material constants which are measurable by means of the introduced ideal experiments. The equations of motion in the frequency domain are obtained in terms of the Fourier transformation. In terms of the equations of motion in frequency domain, the wave speeds and the attenuations for three P waves and one S wave are calculated. The influences of the capillary pressure relaxation coefficient and the saturation of the wetting phase on the velocities and attenuation coefficients for the four waves in the porous medium are discussed in the two numerical examples.

## 2. Kinematics and balance equations

In this paper the porous medium is considered as consisting of a solid phase which is identified by the superscript "s" and two immiscible fluids which are identified by the superscripts "w" (wetting phase) and "n" (non-wetting phase), respectively. In what follows, the variable $\alpha$ is used to denote an individual phase, $\alpha = \mathrm{s, w, n}$, while the variable $f$ refers to either of the two fluid phases, $f = \mathrm{w, n}$. The balance equations of the three phases are formulated in the framework of the generalized mixture theory (Bowen, 1980, 1982). On macroscopic scale each spatial point in the domain of interest is occupied by the material points of each individual phase. The fraction of the microscopic volume occupied by phase $\alpha$ is represented by the volume fraction $n^{(\alpha)}$. In the absence of voids, $\sum_\alpha n^{(\alpha)} = 1$. Each individual phase, therefore, can be characterized by two densities, one of which is true density ($\gamma^{(\alpha)}$) relative to the volume it actually occupies, and the other is bulk density ($\rho^{(\alpha)} = n^{(\alpha)}\gamma^{(\alpha)}$) relative to the total volume occupied by the mixture point. For a porous medium with porosity $\phi$, the fraction of the solid phase $n^{(\mathrm{s})} = 1 - \phi$, where $\phi$ is the porosity of the solid matrix. In the paper, the porosity of the solid matrix is assumed to be a material constant that is not subject to temporal variations. Moreover, the volume fraction of the two fluids can be expressed by using the wetting fluid saturation $S$, with $n^{(\mathrm{w})} = S\phi$ and $n^{(\mathrm{n})} = (1 - S)\phi$.

Generally, the three phases s, w, n move independently in the course of a thermodynamic process. Thus, the material particles of each phase should be characterized by independent initial material position $\mathbf{X}^{(\alpha)}$, and the motion of material point of each phase, therefore, may be expressed as $\mathbf{x}^{(\alpha)}(\mathbf{X}^{(\alpha)}, t)$. However, for linear theory in this paper, the difference between the material coordinate and the Eulerian coordinate is neglected. As a result, the material derivative is reduced to the time derivative at a fixed spatial location. For the linear process under consideration, we assume the following reference equilibrium state

$$\{\gamma^{(f)}, S, \mathbf{E}^{(\mathrm{s})}, \mathbf{v}^{(\alpha)}\} = \{\gamma_0^{(f)}, S_0, \mathbf{0}, \mathbf{0}\},\\ \alpha = \mathrm{s}, \mathrm{w}, \mathrm{n} \tag{1}$$

where $\mathbf{E}^{(\mathrm{s})}$ is the strain tensor for the solid skeleton and $\mathbf{v}^{(\alpha)}$ is the velocity for the three phases. When an infinitesimal disturbance is applied, the porous medium will arrive at a new state specified by the variables

$$\{\gamma^{(f)}, S, \mathbf{E}^{(\mathrm{s})}, \mathbf{v}^{(\alpha)}\} = \{\gamma_0^{(f)} + \Delta\gamma^{(f)}, S_0 + \Delta S, \Delta\mathbf{E}^{(\mathrm{s})}, \Delta\mathbf{v}^{(\alpha)}\}, \quad \alpha = \mathrm{s}, \mathrm{w}, \mathrm{n} \tag{2}$$

For the dynamic process defined by (1), (2), reducing the material derivative to the time derivative at a fixed spatial location, the following mass balance equations for the three phases are obtained

$$\frac{\partial \Delta\gamma^{(\mathrm{s})}}{\partial t} = -\gamma_0^{(\mathrm{s})} \nabla \cdot \mathbf{v}^{(\mathrm{s})} \tag{3a}$$

$$\frac{\partial \Delta\gamma^{(\mathrm{w})}}{\partial t} = -\gamma_0^{(\mathrm{w})} \nabla \cdot \mathbf{v}^{(\mathrm{w})} - \frac{\gamma_0^{(\mathrm{w})}}{S_0} \frac{\partial \Delta S}{\partial t} \tag{3b}$$

$$\frac{\partial \Delta\gamma^{(\mathrm{n})}}{\partial t} = -\gamma_0^{(\mathrm{n})} \nabla \cdot \mathbf{v}^{(\mathrm{n})} + \frac{\gamma_0^{(\mathrm{n})}}{1 - S_0} \frac{\partial \Delta S}{\partial t} \tag{3c}$$

Note that the mass exchange between two components is excluded in (3). Similarly, the linear momentum balance equations for the three phases have the form

$$n_0^{(\alpha)} \gamma_0^{(\alpha)} \frac{\partial \mathbf{v}^{(\alpha)}}{\partial t} - \nabla \cdot \Delta\boldsymbol{\sigma}^{(\alpha)} - n_0^{(\alpha)} \gamma_0^{(\alpha)} \Delta\mathbf{b}^{(\alpha)} = \Delta\overline{\mathbf{T}}^{(\alpha)}, \quad \alpha = \mathrm{s}, \mathrm{w}, \mathrm{n} \tag{4}$$

where $\Delta\boldsymbol{\sigma}^{(\alpha)}$ and $\Delta\mathbf{b}^{(\alpha)}$ are the increments of the stress and the external supply of linear momentum per unit mass for $\alpha$ phase, $\Delta\overline{\mathbf{T}}^{(\alpha)}$ represents the increment of the rate of linear momentum transfer to phase $\alpha$ from the other phases. It is worth noting that since in this paper we concentrate on the isothermal thermodynamics process, the energy balance equation for the porous medium is not needed for the closure of the system.

In order to simplify the notation of the paper, from now on, all the quantities with increment symbol $\Delta$ will be replaced by the corresponding quantities without the symbol $\Delta$ and all the initial quantities associated with the reference equilibrium state will be specified by the subscript 0. Thus, the mass balance equations for the three phases are rewritten as

$$\frac{\partial \gamma^{(\mathrm{s})}}{\partial t} = -\gamma_0^{(\mathrm{s})} \nabla \cdot \mathbf{v}^{(\mathrm{s})} \tag{5a}$$

$$\frac{\partial \gamma^{(\mathrm{w})}}{\partial t} = -\gamma_0^{(\mathrm{w})} \nabla \cdot \mathbf{v}^{(\mathrm{w})} - \frac{\gamma_0^{(\mathrm{w})}}{S_0} \frac{\partial S}{\partial t} \tag{5b}$$

$$\frac{\partial \gamma^{(\mathrm{n})}}{\partial t} = -\gamma_0^{(\mathrm{n})} \nabla \cdot \mathbf{v}^{(\mathrm{n})} + \frac{\gamma_0^{(\mathrm{n})}}{1 - S_0} \frac{\partial S}{\partial t} \tag{5c}$$

The momentum balance equation for each phase is recast in the form

$$n_0^{(\alpha)} \gamma_0^{(\alpha)} \frac{\partial \mathbf{v}^{(\alpha)}}{\partial t} - \nabla \cdot \boldsymbol{\sigma}^{(\alpha)} - n_0^{(\alpha)} \gamma_0^{(\alpha)} \mathbf{b}^{(\alpha)} = \overline{\mathbf{T}}^{(\alpha)}, \quad \alpha = \mathrm{s}, \mathrm{w}, \mathrm{n} \tag{6}$$

The momentum conservation law for the three phase of the porous medium assumes the form

$$\sum_{\alpha} \overline{\mathbf{T}}^{(\alpha)} = 0 \tag{7}$$

Assuming that there is no moment of momentum transfer between the phases, it follows that $\boldsymbol{\sigma}^{(\alpha)} = \boldsymbol{\sigma}^{(\alpha)^{\mathrm{T}}}$. The total mass density, total stress of the three-phase porous medium are defined as follows:

$$\rho_0 = \sum_{\alpha} \rho_0^{(\alpha)} = \sum_{\alpha} n_0^{(\alpha)} \gamma_0^{(\alpha)}, \quad \boldsymbol{\sigma} = \sum_{\alpha} \boldsymbol{\sigma}^{(\alpha)} \tag{8a,b}$$

As usual in poroelasticity, shear stress in fluids is neglected. The relationship between the bulk stress of fluids and the corresponding pore pressures is assumed the form

$$\boldsymbol{\sigma}^{(\mathrm{f})} = -p^{(\mathrm{f})} n_0^{(\mathrm{f})} \mathbf{I}, \quad \mathrm{f} = \mathrm{w}, \mathrm{n} \tag{9}$$

where $p^{(\mathrm{f})}$ denotes the true pressure of the wetting and non-wetting fluid.

## 3. Entropy inequality and constitutive relations

For the linear isothermal dynamic process experienced by the porous medium here, the balance equations presented in the previous part are not sufficient to characterize the process. In order to obtain a closed system of equations, it is necessary to express some physical variables in terms of constitutive functions of the independent variables. In deriving our constitutive relations, following Biot (1956a, 1962) and Hanyga (2004), the porous medium is described in terms of a single total energy. For the porous medium under consideration, $\mathbf{E}^{(\mathrm{s})}$, $\gamma^{(\alpha)}$ ($\alpha = \mathrm{s}, \mathrm{w}, \mathrm{n}$), $S$ are chosen as the independent state variables. Constitutive equations, therefore, must be introduced for the dependent variables: the free energy $A$ for the unit volume of the porous medium and the partial stress $\boldsymbol{\sigma}^{(\alpha)}$ ($\alpha = \mathrm{s}, \mathrm{w}, \mathrm{n}$). Consequently, for the linear theory under consideration, the following constitutive equations for the free energy $A$ and the partial stress $\boldsymbol{\sigma}^{(\alpha)}$ ($\alpha = \mathrm{s}, \mathrm{w}, \mathrm{n}$) are assumed

$$A = \widehat{A}(\mathbf{E}^{(\mathrm{s})}, \gamma^{(\mathrm{w})}, \gamma^{(\mathrm{n})}, S) \tag{10a}$$

$$\boldsymbol{\sigma}^{(\alpha)} = \hat{\boldsymbol{\sigma}}^{(\alpha)}(\mathbf{E}^{(\mathrm{s})}, \gamma^{(\mathrm{w})}, \gamma^{(\mathrm{n})}, S) \tag{10b}$$

where $\widehat{A}$ and $\hat{\boldsymbol{\sigma}}^{(\alpha)}$ represent the function form of the free energy and the partial stress. Note that since the density of solid skeleton can be determined by integrating the mass balance equation of the solid phase in terms of $\mathbf{E}^{(\mathrm{s})}$ and porosity $\phi$ (material constant), $\gamma^{(\mathrm{s})}$ is not explicitly included in above equation.

Following the interpretation of the second law of thermodynamics by Coleman and Noll (1963), all the solutions of the balance equations and constitutive equations must satisfy the Clausius–Duhem inequality. For linear isothermal dynamic process of the porous medium, the entropy inequality assumes the following form (Hanyga and Lu, 2004)

$$-\rho_0 \frac{\partial A}{\partial t} + \sum_{\alpha} \boldsymbol{\sigma}^{(\alpha)} : \mathbf{L}^{(\alpha)} - \overline{\mathbf{T}}^{(\mathrm{w})} \cdot \mathbf{v}^{(\mathrm{w,s})} - \overline{\mathbf{T}}^{(\mathrm{n})} \cdot \mathbf{v}^{(\mathrm{n,s})} \geqslant 0 \tag{11}$$

where $\mathbf{L}^{(\alpha)} = \operatorname{grad} \mathbf{v}^{(\alpha)}$ and $\mathbf{v}^{(f,\mathrm{s})} = \mathbf{v}^{(f)} - \mathbf{v}^{(\mathrm{s})}, f = \mathrm{w}, \mathrm{n}$. By using the constitutive assumptions and the mass balance equations, above inequality leads to

$$\begin{aligned} &\left(\boldsymbol{\sigma}^{(\mathrm{s})} - \rho_0 \frac{\partial A}{\partial \mathbf{E}^{(\mathrm{s})}}\right) : \mathbf{L}^{(\mathrm{s})} + \left(\boldsymbol{\sigma}^{(\mathrm{w})} + \rho_0 \gamma_0^{(\mathrm{w})} \frac{\partial A}{\partial \gamma^{(\mathrm{w})}} \mathbf{I}\right) : \mathbf{L}^{(\mathrm{w})} + \left(\boldsymbol{\sigma}^{(\mathrm{n})} + \rho_0 \gamma_0^{(\mathrm{n})} \frac{\partial A}{\partial \gamma^{(\mathrm{n})}} \mathbf{I}\right) \\ &\quad : \mathbf{L}^{(\mathrm{n})} + \left[\frac{\rho_0 \gamma_0^{(\mathrm{w})}}{S_0} \frac{\partial A}{\partial \gamma^{(\mathrm{w})}} - \frac{\rho_0 \gamma_0^{(\mathrm{n})}}{(1 - S_0)} \frac{\partial A}{\partial \gamma^{(\mathrm{n})}} - \rho_0 \frac{\partial A}{\partial S}\right] \frac{\partial S}{\partial t} - \overline{\mathbf{T}}^{(\mathrm{w})} \cdot \mathbf{v}^{(\mathrm{w,s})} - \overline{\mathbf{T}}^{(\mathrm{n})} \cdot \mathbf{v}^{(\mathrm{n,s})} \geqslant 0 \end{aligned} \tag{12}$$

Since the values of $\mathbf{L}^{(\alpha)}$, $\alpha = \mathrm{s, w, n}$ are not constrained by any balance equations nor constitutive assumptions, it follows that the coefficients of $\mathbf{L}^{(\alpha)}$ in Eq. (12) must vanish

$$\boldsymbol{\sigma}^{(\mathrm{s})} = \rho_0 \frac{\partial A}{\partial \mathbf{E}^{(\mathrm{s})}} \tag{13a}$$

$$\boldsymbol{\sigma}^{(f)} = -\rho_0 \gamma_0^{(f)} \frac{\partial A}{\partial \gamma^{(f)}} \mathbf{I}, \quad p^{(f)} = \frac{\rho_0 \gamma_0^{(f)}}{n_0^{(f)}} \frac{\partial A}{\partial \gamma^{(f)}}, \quad f = \mathrm{w, n} \tag{13b,c}$$

Using Eqs. (12) and (13), the following residual entropy production inequality is obtained

$$\left[ \frac{\rho_0 \gamma_0^{(\mathrm{w})}}{S_0} \frac{\partial A}{\partial \gamma^{(\mathrm{w})}} - \frac{\rho_0 \gamma_0^{(\mathrm{n})}}{(1 - S_0)} \frac{\partial A}{\partial \gamma^{(\mathrm{n})}} - \rho_0 \frac{\partial A}{\partial S} \right] \frac{\partial S}{\partial t} - \overline{\mathbf{T}}^{(\mathrm{w})} \cdot \mathbf{v}^{(\mathrm{w,s})} - \overline{\mathbf{T}}^{(\mathrm{n})} \cdot \mathbf{v}^{(\mathrm{n,s})} \geqslant 0 \tag{14}$$

The time derivative of the saturation $S$ and the relative velocities $\mathbf{v}^{(\mathrm{w,s})}$, $\mathbf{v}^{(\mathrm{n,s})}$ vanish at an equilibrium state. Consequently, inequality (14) is satisfied automatically for any equilibrium state near the reference state. However, for dynamic process, say, wave propagation, in general, the time derivative of the saturation $S$ and relative velocities $\mathbf{v}^{(\mathrm{w,s})}$, $\mathbf{v}^{(\mathrm{n,s})}$ will not vanish. Consequently, the coefficients of $\partial S/\partial t$ and $\mathbf{v}^{(\mathrm{w,s})}$, $\mathbf{v}^{(\mathrm{n,s})}$ in (14) must be chosen to satisfy the inequality. Since linear theory is considered in the paper, the deviation of the perturbed state from the reference equilibrium state is infinitesimal. As a result, capillary pressure hysteresis (Dullien, 1992) is neglected here. Moreover, we shall focus on the low frequency dynamic process of the porous medium. Therefore, the frequency independent drag force model is adopted here (Biot, 1956a). For simplicity, in this paper we neglect the coupling between the capillary effect and the drag force and only reserve the diagonal terms in the quadratic form of $\partial S/\partial t$, $\mathbf{v}^{(\mathrm{w,s})}$, $\mathbf{v}^{(\mathrm{n,s})}$ determined by (14). Thus, the porosity relaxation equation and drag force between the fluids and the solid are assumed in the following forms

$$\frac{\rho_0 \gamma_0^{(\mathrm{w})}}{S_0} \frac{\partial A}{\partial \gamma^{(\mathrm{w})}} - \frac{\rho_0 \gamma_0^{(\mathrm{n})}}{(1 - S_0)} \frac{\partial A}{\partial \gamma^{(\mathrm{n})}} - \rho_0 \frac{\partial A}{\partial S} = K_{\mathrm{ds}} \frac{\partial S}{\partial t} \tag{15a}$$

$$\overline{\mathbf{T}}^{(\mathrm{w})} = -b_{\mathrm{w}} \mathbf{v}^{(\mathrm{w,s})} = -b_{\mathrm{w}} (\mathbf{v}^{(\mathrm{w})} - \mathbf{v}^{(\mathrm{s})}) \tag{15b}$$

$$\overline{\mathbf{T}}^{(\mathrm{n})} = -b_{\mathrm{n}} \mathbf{v}^{(\mathrm{n,s})} = -b_{\mathrm{n}} (\mathbf{v}^{(\mathrm{n})} - \mathbf{v}^{(\mathrm{s})}) \tag{15c}$$

where the parameter $K_{\mathrm{ds}}$ is the porosity relaxation coefficient and accounts for the relaxation process regarding the saturation variation; $b_{\mathrm{w}}$, $b_{\mathrm{n}}$ are the parameters accounting for the drag forces between the solid skeleton and pore fluids. It follows from (15), in order to guarantee the inequality (14), the coefficients $K_{\mathrm{ds}}$, $b_{\mathrm{w}}$, $b_{\mathrm{n}}$ should be non-negative. There are two mechanisms accounting for the attenuation of the two-fluid saturated porous medium in our model: the common drag force mechanism and the present capillary pressure relaxation mechanism. It will be shown later the capillary pressure relaxation is an important attenuation mechanism for the P waves of the porous medium. Clearly, the entropy inequality (14) is satisfied automatically if the dynamic process is restricted by (15). Moreover, the parameter $K_{\mathrm{ds}}$ has the dimension of $SN/L^2$ and drag force parameters $b_{\mathrm{w}}$, $b_{\mathrm{n}}$ are given by the following expression

$$b_f = \frac{n_0^{(f)^2} \eta^{(f)}}{k k_{\mathrm{r}}^{(f)}}, \quad f = \mathrm{w, n} \tag{16}$$

where $k$ is the intrinsic permeability of the porous medium, $k_{\mathrm{r}}^{(f)}$ is the relative permeability for fluid which is dependent on the saturation of the fluid (Dullien, 1992), $\eta^{(f)}$ is the viscosity of the fluid. Note that the frequency independent drag force model used here only holds for the low frequency range (Biot, 1956a), as for the high frequency, drag force models are frequency dependent (Biot, 1956b; Johnson et al., 1987; Pride et al., 1993).

The following nine momentum balance equations are obtained by substituting (15b) and (15c) into the linear momentum balance equation (6) for the three phases

$$\nabla \cdot \boldsymbol{\sigma}^{(s)} + n_0^{(s)}\gamma_0^{(s)}\mathbf{b}^{(s)} + b_w(\mathbf{v}^{(w)} - \mathbf{v}^{(s)}) + b_n(\mathbf{v}^{(n)} - \mathbf{v}^{(s)}) = n_0^{(s)}\gamma_0^{(s)}\frac{\partial \mathbf{v}^{(s)}}{\partial t} \tag{17a}$$

$$-\nabla \cdot (n_0^{(f)}p^{(f)}\mathbf{I}) + n_0^{(f)}\gamma_0^{(f)}\mathbf{b}^{(f)} - b_f(\mathbf{v}^{(f)} - \mathbf{v}^{(s)}) = n_0^{(f)}\gamma_0^{(f)}\frac{\partial \mathbf{v}^{(f)}}{\partial t}, \quad f = \mathrm{w,n} \tag{17b}$$

Note that since our theory is based on the first principle of the continuum mechanics and the thermodynamics, consequently, the intuition-based mass coupling is neglected in our momentum balance equations. Using Eq. (13), Eq. (15a) is rewritten as

$$\phi(p^{(w)} - p^{(n)}) - \rho_0\frac{\partial A}{\partial S} = K_{ds}\dot{S} \tag{18}$$

where a dot over a variable denotes the derivative with respect to time.

## 4. Formulation of the linear model for the porous medium

Based on the theory of above section, a more specific linear model for a porous medium saturated by two fluids will be developed in this section. For the linear porous medium under consideration here, the free energy for a unit volume porous medium can be expressed by the following quadratic form

$$\begin{aligned}\rho_0 A = {} & \frac{1}{2}\mathbf{E}^{(s)} : \mathbf{D} : \mathbf{E}^{(s)} + \frac{1}{2}n_0^{(w)}M_{ww}\left(\frac{\gamma^{(w)}}{\gamma_0^{(w)}}\right)^2 + \frac{1}{2}n_0^{(n)}M_{nn}\left(\frac{\gamma^{(n)}}{\gamma_0^{(n)}}\right)^2 + \frac{1}{2}n_0^{(w)}n_0^{(n)}K_{cp}S^2 \\ & + n_0^{(w)}n_0^{(n)}M_{wn}\left(\frac{\gamma^{(w)}}{\gamma_0^{(w)}}\right)\left(\frac{\gamma^{(n)}}{\gamma_0^{(n)}}\right) - n_0^{(w)}M_{sw}(\mathbf{E}^{(s)} : \mathbf{I})\left(\frac{\gamma^{(w)}}{\gamma_0^{(w)}}\right) - n_0^{(n)}M_{sn}(\mathbf{E}^{(s)} : \mathbf{I})\left(\frac{\gamma^{(n)}}{\gamma_0^{(n)}}\right) \\ & + n_0^{(w)}n_0^{(n)}M_{cs}(\mathbf{E}^{(s)} : \mathbf{I})S - n_0^{(w)^2}n_0^{(n)}M_{cw}\left(\frac{\gamma^{(w)}}{\gamma_0^{(w)}}\right)S - n_0^{(w)}n_0^{(n)^2}M_{cn}\left(\frac{\gamma^{(n)}}{\gamma_0^{(n)}}\right)S\end{aligned} \tag{19}$$

where $\mathbf{D} = \lambda_s\mathbf{I} \otimes \mathbf{I} + 2\mu_s\mathbf{I}_4$, $\mathbf{I}$ is the second-order isotropic tensor with component $\delta_{ij}$ ($\delta_{ij}$ is the Kronecker delta), $\mathbf{I}_4$ is the fourth-order isotropic tensor with component $(\mathbf{I}_4)_{ijkl} = 1/2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, $\mu_s$, $\lambda_s$ are elastic parameters for the solid skeleton. In expression (19), the first three terms denote the energy of three bulk phases: the solid phase, the wetting phase and the non-wetting phase; the fourth and the fifth term denote the interface energy between the two fluids; the sixth and the seventh term represent the interface energy between the solid and the two fluids; while the last three terms denote the common contact lines energy of the three phases (Gray and Hassanizadeh, 1991). Compared with the energy of the three bulk phases and the energy of the interfaces, the energy due to the common contact lines is assumed to be negligible. As a result, in the following derivation, we set $M_{cs} = M_{cw} = M_{cn} = 0$. Note that the minus symbol in the above equation is due to the fact $-\gamma^{(f)}/\gamma_0^{(f)}$, $f = \mathrm{w,n}$ denotes the volume variation of the $f$-phase and the volume fraction coefficient before each term denote the volume of the fluid related to the energy represented by the term.

Integration of (5) leads to the following mass balance equations

$$\gamma^{(s)} = -\gamma_0^{(s)}\nabla \cdot \mathbf{u}^{(s)} \tag{20a}$$

$$\gamma^{(\mathrm{w})} = -\gamma_0^{(\mathrm{w})}\nabla\cdot\mathbf{u}^{(\mathrm{w})} - \frac{\gamma_0^{(\mathrm{w})}}{S_0}S \tag{20b}$$

$$\gamma^{(\mathrm{n})} = -\gamma_0^{(\mathrm{n})}\nabla\cdot\mathbf{u}^{(\mathrm{n})} + \frac{\gamma_0^{(\mathrm{n})}}{1-S_0}S \tag{20c}$$

Using (13), (19) and (20), the stress for the solid skeleton has the form

$$\boldsymbol{\sigma}^{(\mathrm{s})} = \mathbf{D}:\mathbf{E}^{(\mathrm{s})} + n_0^{(\mathrm{w})}M_{\mathrm{sw}}e_{\mathrm{w}}\mathbf{I} + n_0^{(\mathrm{n})}M_{\mathrm{sn}}e_{\mathrm{n}}\mathbf{I} + N_{\mathrm{ss}}S\mathbf{I} \tag{21}$$

where $N_{\mathrm{ss}} = \phi(M_{\mathrm{sw}} - M_{\mathrm{sn}})$ and $e_{\mathrm{w}} = \nabla\cdot u^{(\mathrm{w})}$, $e_{\mathrm{n}} = \nabla\cdot\mathbf{u}^{(\mathrm{n})}$. The pore pressures for the two fluids can be calculated in a similar way

$$p^{(\mathrm{w})} = -M_{\mathrm{sw}}e_{\mathrm{s}} - M_{\mathrm{ww}}e_{\mathrm{w}} - n_0^{(\mathrm{n})}M_{\mathrm{wn}}e_{\mathrm{n}} - N_{\mathrm{ws}}S \tag{22a}$$

$$p^{(\mathrm{n})} = -M_{\mathrm{sn}}e_{\mathrm{s}} - n_0^{(\mathrm{w})}M_{\mathrm{wn}}e_{\mathrm{w}} - M_{\mathrm{nn}}e_{\mathrm{n}} + N_{\mathrm{ns}}S \tag{22b}$$

where $N_{\mathrm{ws}} = M_{\mathrm{ww}}/S_0 - \phi M_{\mathrm{wn}}$, $N_{\mathrm{ns}} = M_{\mathrm{nn}}/(1-S_0) - \phi M_{\mathrm{wn}}$ and $e_{\mathrm{s}} = \nabla\cdot\mathbf{u}^{(\mathrm{s})}$. In terms of (18) and (19), the relaxation equation for the capillary pressure is obtained

$$\phi(p^{(\mathrm{w})} - p^{(\mathrm{n})}) - n_0^{(\mathrm{w})}n_0^{(\mathrm{n})}K_{\mathrm{cp}}S = K_{\mathrm{ds}}\dot{S} \tag{23}$$

In equilibrium, the derivative of the saturation vanishes and Eq. (23) implies the increment of the capillary pressure is related to the saturation increment. In a dynamic process, the increment of the capillary pressure is related to the derivative of the saturation as well as the saturation increment. Note that (23) also serves as the closure equation for the system. Also, Eq. (23) shows that for the equilibrium state, the capillary pressure of the porous medium varies linearly with the increment of the saturation. This is the linear property of our theory. In effect, the equilibrium expression of Eq. (23) is similar to the differentiation of the Leverett type capillary pressure relation (Leverett, 1941).

As is well known, the free energy of the porous medium must be non-negative, so the quadratic form (19) should be positive definite. The constraints on the material parameters by the positive definite condition are as follows:

$$K_{\mathrm{s}} > 0,\quad \mu_{\mathrm{s}} > 0,\quad M_{\mathrm{ww}} > 0,\quad M_{\mathrm{nn}} > 0,\quad K_{\mathrm{cp}} > 0,$$

$$\begin{bmatrix} K_{\mathrm{s}} & -n_0^{(\mathrm{w})}M_{\mathrm{sw}} \\ -n_0^{(\mathrm{w})}M_{\mathrm{sw}} & n_0^{(\mathrm{w})}M_{\mathrm{ww}} \end{bmatrix} > 0,\quad \begin{bmatrix} K_{\mathrm{s}} & -n_0^{(\mathrm{w})}M_{\mathrm{sw}} & -n_0^{(\mathrm{n})}M_{\mathrm{sn}} \\ -n_0^{(\mathrm{w})}M_{\mathrm{sw}} & n_0^{(\mathrm{w})}M_{\mathrm{ww}} & n_0^{(\mathrm{w})}n_0^{(\mathrm{n})}M_{\mathrm{wn}} \\ -n_0^{(\mathrm{n})}M_{\mathrm{sn}} & n_0^{(\mathrm{w})}n_0^{(\mathrm{n})}M_{\mathrm{wn}} & n_0^{(\mathrm{n})}M_{\mathrm{nn}} \end{bmatrix} > 0 \tag{24a–g}$$

where $K = \lambda_{\mathrm{s}} + (2/3)\mu_{\mathrm{s}}$.

Summarily, the governing equations for the porous medium here are constitutive relations (21) and (22), the closure equation (23) and the momentum balance equation (17). These equations are sufficient for solving the displacements, stress, pore pressures and the saturation of the system. It is worth noting that based on our two-fluid theory, one fluid theory (Biot's theory) can be recovered by letting the volume fraction of the wetting phase or the non-wetting phase tend to zero or by letting the material parameters of the two fluids be identical if the mass coupling in Biot's theory is neglected.

## 5. Estimation of the parameters involved in the linear model for the porous medium

Nine parameters $K_{\mathrm{cp}}$, $K_{\mathrm{ds}}$, $\lambda_{\mathrm{s}}$, $\mu_{\mathrm{s}}$, $M_{\mathrm{ww}}$, $M_{\mathrm{nn}}$, $M_{\mathrm{wn}}$, $M_{\mathrm{sw}}$, $M_{\mathrm{sn}}$ in the proposed model have to be determined. These parameters are only indirectly measurable. Thus, in order to determine these parameters, the relationships between these parameters and the measurable parameters must be established. Note that it is assumed that the parameters such as permeability, relative permeability and viscosity of the fluids are known in advance and they can be measured by common experiments (Dullien, 1992). Consequently, their evaluation will not be discussed here.

In this paper, six experiments and ten phenomenological parameters are introduced to determine the nine material parameters. Firstly, the matric suction test (Fredlund and Rahardjo, 1993) and the corresponding phenomenological parameter—the specific saturation capacity—are introduced to evaluate the parameter $K_{\mathrm{cp}}$. Secondly, a harmonic dynamic experiment concerning the capillary pressure is used to determine the capillary pressure relaxation coefficient $K_{\mathrm{ds}}$ in (23) and the phase difference between the capillary pressure increment and the saturation increment is used as the phenomenological parameter. Moreover, for the two elastic parameters $\mu_{\mathrm{s}}$, $K_{\mathrm{s}}$, a shear test and a drained compression test for the porous medium sample are used, while the shear modulus $\mu_{\mathrm{d}}$ and the drained bulk modulus $K_{\mathrm{dn}}$ for the two-fluid saturated sample are used as two phenomenological parameters. Finally, for the other five parameters $M_{\mathrm{ww}}$, $M_{\mathrm{nn}}$, $M_{\mathrm{wn}}$, $M_{\mathrm{sw}}$, $M_{\mathrm{sn}}$, the undrained and the quasi-unjacketed test for the porous medium will be used. For the undrained test of the porous medium, the undrained modulus $K_{\mathrm{ud}}$, the Skempton coefficients $B_{\mathrm{w}}$, $B_{\mathrm{n}}$ (Skempton, 1954) for the wetting and non-wetting phase are used as three phenomenological parameters, while for quasi-unjacketed test (Biot and Willis, 1957), the quasi-unjacketed bulk modulus $K_{\mathrm{uj}}$ and the fluid content coefficients $\beta_{\mathrm{w}}$, $\beta_{\mathrm{n}}$ for the wetting and non-wetting phase are chosen as the other three phenomenological parameters. The relations between the six phenomenological parameters $K_{\mathrm{ud}}$, $B_{\mathrm{w}}$, $B_{\mathrm{n}}$, $K_{\mathrm{uj}}$, $\beta_{\mathrm{w}}$, $\beta_{\mathrm{n}}$ and five parameters to be determined will be established in this section.

Now we proceed to evaluate the nine material parameters. We begin with the matric suction test (Fredlund and Rahardjo, 1993). When a small disturbance is applied to the test sample at a reference equilibrium state, it will deviate from the reference equilibrium state. After a period of time, the test sample will arrive at a new equilibrium state. Consequently, in terms of (23), for the new equilibrium state, the following equation holds

$$\phi(p^{(\mathrm{w})} - p^{(\mathrm{n})}) - n_0^{(\mathrm{w})} n_0^{(\mathrm{n})} K_{\mathrm{cp}} S = 0 \tag{25}$$

If introducing the specific saturation capacity $\Gamma_{\mathrm{c}}$, the relation between the variation of the capillary pressure and that of the saturation of wetting phase has the form

$$p^{(\mathrm{n})} - p^{(\mathrm{w})} = S/\Gamma_{\mathrm{c}} \tag{26}$$

where $\Gamma_{\mathrm{c}}$ can be measured by the matric suction test (Fredlund and Rahardjo, 1993). Using (25) and (26), $K_{\mathrm{cp}}$ is obtained

$$K_{\mathrm{cp}} = -\frac{\phi}{n_0^{(\mathrm{w})} n_0^{(\mathrm{n})} \Gamma_{\mathrm{c}}} \tag{27}$$

In order to determine the parameter $K_{\mathrm{ds}}$ in (23), we introduce a dynamic experiment. Suppose the volume of the solid skeleton and the pressure of the wetting phase are kept constant. The wetting phase can flow in and out through a semi-permeable tube which connects the test sample to a reservoir having the same fixed pressure as the wetting phase. The volume variation of the wetting phase can be measured by a volume meter installed inside the semi-permeable tube. The pressure of the non-wetting phase has the following harmonic expression

$$p^{(\mathrm{n})} = -p_{\mathrm{a}} \mathrm{e}^{\mathrm{i}\omega t} \tag{28}$$

Then, according to eq. (23), the saturation has the following form

$$S = \frac{\phi p_{\mathrm{a}} \mathrm{e}^{\mathrm{i}\omega t}}{n_0^{(\mathrm{w})} n_0^{(\mathrm{n})} K_{\mathrm{cp}} + \mathrm{i}\omega K_{\mathrm{ds}}} = \frac{\phi p_{\mathrm{a}} \mathrm{e}^{\mathrm{i}(\omega t - \alpha_0)}}{\sqrt{(n_0^{(\mathrm{w})} n_0^{(\mathrm{n})} K_{\mathrm{cp}})^2 + (\omega K_{\mathrm{ds}})^2}} \tag{29}$$

where $\alpha_0 = \mathrm{Arc}\tan[\omega K_{ds}/(n_0^{(w)} n_0^{(n)} K_{cp})]$ is the phase difference between the increment of capillary pressure and the increment of the saturation. Assuming the real input signal of $p^{(n)}$ is

$$p^{(n)} = -p_a \cos(\omega t) \tag{30}$$

in view of (29), the real saturation output is given by

$$S = \frac{\phi p_a}{\sqrt{(n_0^{(w)} n_0^{(n)} K_{cp})^2 + (\omega K_{ds})^2}} \cos(\omega t - \alpha_0) \tag{31}$$

Because the volume variation of the wetting phase is in phase with the saturation of the sample, the phase of the saturation can be evaluated by measuring the phase of the volume variation of the wetting fluid. After obtaining the phase difference $\alpha_0$, the parameter $K_{ds}$ can be calculated using the formula

$$K_{ds} = \frac{(n_0^{(w)} n_0^{(n)} K_{cp}) \tan(\alpha_0)}{\omega} \tag{32}$$

As mentioned above, $\mu_s$, $\lambda_s$ may be determined by shear and compression sample tests. By a shear test performed on a two-fluid saturated porous sample, one has

$$\mu_s = \mu_d \tag{33}$$

where $\mu_d$ is the shear modulus of the sample.

The experimental model for the drained test is shown in Fig. 1. In Fig. 1, the porous medium sample is enclosed by an impermeable jacket and is connected to the wetting and non-wetting fluid reservoirs by two semi-permeable tubes. The semi-permeable tube of the wetting phase is impermeable to the non-wetting phase, while the semi-permeable tube of the non-wetting phase is impermeable to the wetting phase. The pressures of the two reservoirs are under control. For Fig. 1 experiment model, if let $p^{(w)} = p^{(n)} = 0$, then the drained bulk modulus ($K_{dn} = -\Delta p/e_s$) can be obtained by measuring the pressure increment and the volume strain of the solid skeleton. By using Eqs. (21) and (22), $K_s$ is given by

$$K_s = K_{dn} + \frac{n_0^{(w)} M_{nn} M_{sw}^2 + n_0^{(n)} M_{ww} M_{sn}^2 - 2 n_0^{(w)} n_0^{(n)} M_{sw} M_{sn} M_{wn}}{M_{ww} M_{nn} - n_0^{(w)} n_0^{(n)} M_{wn}^2} \tag{34}$$

where $\Delta p = -(\mathbf{I}{:}\boldsymbol{\sigma})/3$. It follows from (34) that $K_s$ depends on $M_{ww}$, $M_{nn}$, $M_{sw}$, $M_{sn}$, $M_{wn}$ as well as $K_{dn}$.

Four parameters $K_{cp}$, $K_{ds}$, $\mu_s$, $K_s$ have been determined by four experiments. In order to evaluate the remaining five parameters: $M_{ww}$, $M_{nn}$, $M_{sw}$, $M_{sn}$, $M_{wn}$, two additional experiments will be introduced, i.e., an undrained test and a quasi-unjacketed test. For the undrained test, no local diffusion is permitted

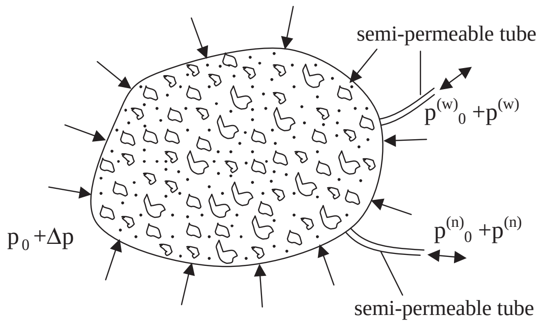


Fig. 1. Porous medium enclosed by an impermeable jacket and connected to the wetting and non-wetting fluid reservoirs by two semi-permeable tubes.

and no fluid is allowed to flow in or out through the boundary of the sample, which is equivalent to closing the two semi-permeable tubes (Fig. 1) connecting to the two reservoirs. Formally, the undrained test is defined by the constraint

$$e_{\mathrm{s}} = e_{\mathrm{w}} = e_{\mathrm{n}} \tag{35}$$

The undrained test here is assumed to be a quasi-static process, which implies that in the test the sample starts from a reference equilibrium state and arrives at a new equilibrium state through a quasi-static process. As a result, Eq. (25) holds for the test. For undrained test three variables are measurable in the experiment, i.e., undrained bulk modulus $K_{\mathrm{ud}}$, the Skempton pore pressure coefficients $B_{\mathrm{w}}$, $B_{\mathrm{n}}$ for the wetting and the non-wetting phase. The undrained bulk modulus $K_{\mathrm{ud}}$ and the Skempton pore pressure coefficients $B_{\mathrm{w}}$, $B_{\mathrm{n}}$ (Skempton, 1954) are defined as follows:

$$K_{\mathrm{ud}} := -\frac{\Delta p}{e_{\mathrm{s}}}, \quad B_{\mathrm{w}} := \frac{p^{(\mathrm{w})}}{\Delta p}, \quad B_{\mathrm{n}} := \frac{p^{(\mathrm{n})}}{\Delta p} \tag{36a,b,c}$$

where $\Delta p$ is the confining pressure increment of the sample. For the undrained test here $\Delta p = -(\mathbf{I}{:}\boldsymbol{\sigma})/3$, where $\boldsymbol{\sigma}$ is the total stress defined in (8). Using (36a,b,c), (25) can be recast in the form

$$S = \frac{\phi \Delta p (B_{\mathrm{w}} - B_{\mathrm{n}})}{n_0^{(\mathrm{w})} n_0^{(\mathrm{n})} K_{\mathrm{cp}}} \tag{37}$$

Calculating the trace of (21) and dividing the resulting equation by 3, one has

$$\sigma_{jj}^{(\mathrm{s})}/3 = K_{\mathrm{s}} e_{\mathrm{s}} + n_0^{(\mathrm{w})} M_{\mathrm{sw}} e_{\mathrm{w}} + n_0^{(\mathrm{n})} M_{\mathrm{sn}} e_{\mathrm{n}} + N_{\mathrm{ss}} S \tag{38}$$

Using (37), (35) and (36a), (38) can be rewritten as

$$\left(1 - n_0^{(\mathrm{w})} B_{\mathrm{w}} - n_0^{(\mathrm{n})} B_{\mathrm{n}}\right) K_{\mathrm{ud}} = K_{\mathrm{s}} + n_0^{(\mathrm{w})} M_{\mathrm{sw}} + n_0^{(\mathrm{n})} M_{\mathrm{sn}} - \frac{N_{\mathrm{ss}} \phi (B_{\mathrm{w}} - B_{\mathrm{n}}) K_{\mathrm{ud}}}{n_0^{(\mathrm{w})} n_0^{(\mathrm{n})} K_{\mathrm{cp}}} \tag{39}$$

Similarly, using (37), (35), (36), Eq. (22) has the following form

$$B_{\mathrm{w}} K_{\mathrm{ud}} = M_{\mathrm{sw}} + M_{\mathrm{ww}} + n_0^{(\mathrm{n})} M_{\mathrm{wn}} - \frac{N_{\mathrm{ws}} \phi (B_{\mathrm{w}} - B_{\mathrm{n}}) K_{\mathrm{ud}}}{n_0^{(\mathrm{w})} n_0^{(\mathrm{n})} K_{\mathrm{cp}}} \tag{40a}$$

$$B_{\mathrm{n}} K_{\mathrm{ud}} = M_{\mathrm{sn}} + n_0^{(\mathrm{w})} M_{\mathrm{wn}} + M_{\mathrm{nn}} + \frac{N_{\mathrm{ns}} \phi (B_{\mathrm{w}} - B_{\mathrm{n}}) K_{\mathrm{ud}}}{n_0^{(\mathrm{w})} n_0^{(\mathrm{n})} K_{\mathrm{cp}}} \tag{40b}$$

We shall now consider the quasi-unjacketed test. The unjacketed test was first introduced by Biot and Willis (1957) for one-fluid-saturated porous medium. Here, we extend the experiment to the porous medium saturated by two fluids. However, our quasi-unjacketed test is different from the Biot's one. The experiment model for the quasi-unjacketed test can also be demonstrated by Fig. 1. The quasi-unjacketed test of the porous medium can be achieved by letting the pore pressure increments of the two fluids equal to the increment of the confining pressure, i.e., $p^{(\mathrm{w})} = p^{(\mathrm{n})} = \Delta p$. The pressures of the two fluids can be controlled by two semi-permeable tubes connected to the two reservoirs. In this experiment, three quantities can be calculated, i.e., the quasi-unjacketed bulk modulus $K_{\mathrm{uj}}$ as well as the fluid content coefficients $\beta_{\mathrm{w}}$, $\beta_{\mathrm{n}}$ for the wetting and the non-wetting phase. The quasi-unjacketed bulk modulus can be obtained by measuring the increment of the confining pressure and the volume strain of the solid phase, while the fluid content coefficients can be obtained by measuring the increment of the confining pressure and the volume of the fluids flowing in or out the sample through the two semi-permeable tubes.

As in the case of the undrained test, we assume that the quasi-unjacketed test here is also a quasi-static process. Thus, Eq. (25) holds for the quasi-unjacketed test too. Since the increment of the pressures for the two fluids are always equal, consequently, in view of (25), it follows that the increment of the saturation of the wetting phase will vanish ($S = 0$). Therefore, the quasi-unjacketed bulk modulus $K_{uj}$ and the fluid content coefficients $\beta_w$, $\beta_n$ are defined as follows:

$$K_{uj} := -\frac{\Delta p}{e_s}, \quad \beta_w := \frac{n_0^{(w)}(e_s - e_w)}{\Delta p}, \quad \beta_n := \frac{n_0^{(n)}(e_s - e_n)}{\Delta p} \tag{41a,b,c}$$

where $\Delta p$ is the pressure increment of the test fluid. The fluid content coefficients $(\beta_w, \beta_n)$ represent the fluids volume increments in a unit volume of the porous medium due to a unit increment of the confining pressure during the quasi-unjacketed test. It is worth noting when define $\beta_w$, $\beta_n$, (20b) and (20c) as well as the condition $S = 0$ have been used here. In terms of (41), the condition of $S = 0$ and Eqs. (38) and (22), the following equations are obtained

$$n_0^{(s)} K_{uj} = K_s + n_0^{(w)} M_{sw}\left(1 + \frac{\beta_w}{n_0^{(w)}} K_{uj}\right) + n_0^{(n)} M_{sn}\left(1 + \frac{\beta_n}{n_0^{(n)}} K_{uj}\right) \tag{42a}$$

$$K_{uj} = M_{sw} + M_{ww}\left(1 + \frac{\beta_w}{n_0^{(w)}} K_{uj}\right) + n_0^{(n)} M_{wn}\left(1 + \frac{\beta_n}{n_0^{(n)}} K_{uj}\right) \tag{42b}$$

$$K_{uj} = M_{sn} + n_0^{(w)} M_{wn}\left(1 + \frac{\beta_w}{n_0^{(w)}} K_{uj}\right) + M_{nn}\left(1 + \frac{\beta_n}{n_0^{(n)}} K_{uj}\right) \tag{42c}$$

Using (34), (39), (40) and arbitrary two equations of (42), the five parameters $M_{ww}$, $M_{nn}$, $M_{sw}$, $M_{sn}$, $M_{wn}$ as well as $K_s$ can be solved analytically. Since Eq. (34) is non-linear, the expressions of the five parameters $M_{ww}$, $M_{nn}$, $M_{sw}$, $M_{sn}$, $M_{wn}$ and $K_s$ are lengthy. Therefore, they will not be listed here.

It is worth noting that the six parameters $K_{ud}$, $B_w$, $B_n$, $K_{uj}$, $\beta_w$, $\beta_n$ measured in the undrained and the quasi-unjacketed test are not independent. If using (39),(40) and (42b), (42c) to calculate the parameters $M_{ww}$, $M_{nn}$, $M_{sw}$, $M_{sn}$, $M_{wn}$ and substituting the obtained $M_{sn}$, $M_{sn}$ into (42a), the following relation between the six parameters is obtained

$$K_{uj} = \frac{K_{ud}}{1 - K_{ud}(B_w \beta_w + B_n \beta_n)} \tag{43}$$

Consequently, when do experiments to measure $K_{ud}$, $B_w$, $B_n$, $K_{uj}$, $\beta_w$, $\beta_n$, only five of them need to be measured, while the remaining one can be calculated by (43).

## 6. Harmonic waves in the homogeneous porous medium

### *6.1. Frequency domain formulation of the problem*

In this section, the theory proposed in the paper will be use to analyze harmonic plane waves propagating in an unbounded homogeneous porous medium. Wave speeds and attenuations of different wave modes are calculated. For harmonic motion, the closure equation (23) can be used to eliminate the saturation from the constitutive relations of the stress and the pore pressures. Substitution of (22) into (23) and application of the Fourier transformation to the resulting equation yield the expression for the saturation in the frequency domain

$$\tilde{S} = Q_{ss}\tilde{e}_s + Q_{sw}\tilde{e}_w + Q_{sn}\tilde{e}_n \tag{44}$$

where a tilde over a variable denotes the Fourier transformation and

$$Q_{ss} = \frac{M_{sn} - M_{sw}}{M_{ww}/S_0 + M_{nn}/(1-S_0) - 2M_{wn}\phi + n_0^{(w)}(1-S_0)K_{cp} + i\omega K_{ds}/\phi} \tag{45a}$$

$$Q_{sw} = \frac{n_0^{(w)}M_{wn} - M_{ww}}{M_{ww}/S_0 + M_{nn}/(1-S_0) - 2M_{wn}\phi + n_0^{(w)}(1-S_0)K_{cp} + i\omega K_{ds}/\phi} \tag{45b}$$

$$Q_{sn} = \frac{M_{nn} - n_0^{(n)}M_{wn}}{M_{ww}/S_0 + M_{nn}/(1-S_0) - 2M_{wn}\phi + n_0^{(w)}(1-S_0)K_{cp} + i\omega K_{ds}/\phi} \tag{45c}$$

Application of the Fourier transformation to (21) and substitution of (44) into the resulting equation give the stress for the solid skeleton in the frequency domain

$$\tilde{\boldsymbol{\sigma}}^{(s)} = \lambda_s \tilde{e}_s \mathbf{I} + 2\mu_s \widetilde{\mathbf{E}}^{(s)} + R_{ss}\tilde{e}_s\mathbf{I} + R_{sw}\tilde{e}_w\mathbf{I} + R_{sn}\tilde{e}_n\mathbf{I} \tag{46}$$

where $R_{ss} = N_{ss}Q_{ss}$, $R_{sw} = N_{ss}Q_{sw} + n_0^{(w)}M_{sw}$, $R_{sn} = N_{ss}Q_{sn} + n_0^{(n)}M_{sn}$. Similarly, application of the Fourier transformation to (22) and substitution of (44) into the resulting equation give the pressures for the two fluids

$$\tilde{p}^{(w)} = -R_{ws}\tilde{e}_s - R_{ww}\tilde{e}_w - R_{wn}\tilde{e}_n \tag{47a}$$

$$\tilde{p}^{(n)} = -R_{ns}\tilde{e}_s - R_{nw}\tilde{e}_w - R_{nn}\tilde{e}_n \tag{47b}$$

where $R_{ws} = N_{ws}Q_{ss} + M_{sw}$, $R_{ww} = N_{ws}Q_{sw} + M_{ww}$, $R_{wn} = N_{ws}Q_{sn} + n_0^{(n)}M_{wn}$, $R_{ns} = -N_{ns}Q_{ss} + M_{sn}$, $R_{nw} = -N_{ns}Q_{sw} + n_0^{(w)}M_{wn}$, $R_{nn} = -N_{ns}Q_{sn} + M_{nn}$. Applying Fourier transformation to (17) and substituting (46), (47) into the resulting equations yields the motion of equations for the three phases in the frequency domain

$$\begin{aligned}&[(\lambda_s + \mu_s) + R_{ss}]\nabla\nabla\cdot\tilde{\mathbf{u}}^{(s)} + \mu_s\nabla\cdot\nabla\tilde{\mathbf{u}}^{(s)} + R_{sw}\nabla\nabla\cdot\tilde{\mathbf{u}}^{(w)} + R_{sn}\nabla\nabla\cdot\tilde{\mathbf{u}}^{(n)} + i\omega b_w(\tilde{\mathbf{u}}^{(w)} - \tilde{\mathbf{u}}^{(s)})\\ &\quad + i\omega b_n(\tilde{\mathbf{u}}^{(n)} - \tilde{\mathbf{u}}^{(s)}) = -n_0^{(s)}\omega^2\gamma_0^{(s)}\tilde{\mathbf{u}}^{(s)}\end{aligned} \tag{48a}$$

$$n_0^{(w)}(R_{ws}\nabla\nabla\cdot\tilde{\mathbf{u}}^{(s)} + R_{ww}\nabla\nabla\cdot\tilde{\mathbf{u}}^{(w)} + R_{wn}\nabla\nabla\cdot\tilde{\mathbf{u}}^{(n)}) - i\omega b_w(\tilde{\mathbf{u}}^{(w)} - \tilde{\mathbf{u}}^{(s)}) = -n_0^{(w)}\omega^2\gamma_0^{(w)}\tilde{\mathbf{u}}^{(w)} \tag{48b}$$

$$n_0^{(n)}(R_{ns}\nabla\nabla\cdot\tilde{\mathbf{u}}^{(s)} + R_{nw}\nabla\nabla\cdot\tilde{\mathbf{u}}^{(w)} + R_{nn}\nabla\nabla\cdot\tilde{\mathbf{u}}^{(n)}) - i\omega b_n(\tilde{\mathbf{u}}^{(n)} - \tilde{\mathbf{u}}^{(s)}) = -n_0^{(n)}\omega^2\gamma_0^{(n)}\tilde{\mathbf{u}}^{(n)} \tag{48c}$$

Applying the divergence operator on the both sides of (48a), (48b), (48c), the following equations for the dilatations of the three phases are obtained

$$\begin{aligned}&[(\lambda_s + 2\mu_s) + R_{ss}]\nabla\cdot\nabla\tilde{e}_s + R_{sw}\nabla\cdot\nabla\tilde{e}_w + R_{sn}\nabla\cdot\nabla\tilde{e}_n + i\omega b_w(\tilde{e}_w - \tilde{e}_s) + i\omega b_n(\tilde{e}_n - \tilde{e}_s)\\ &\quad = -n_0^{(s)}\omega^2\gamma_0^{(s)}\tilde{e}_s\end{aligned} \tag{49a}$$

$$n_0^{(w)}(R_{ws}\nabla\cdot\nabla\tilde{e}_s + R_{ww}\nabla\cdot\nabla\tilde{e}_w + R_{wn}\nabla\cdot\nabla\tilde{e}_n) - i\omega b_w(\tilde{e}_w - \tilde{e}_s) = -n_0^{(w)}\omega^2\gamma_0^{(w)}\tilde{e}_w \tag{49b}$$

$$n_0^{(n)}(R_{ns}\nabla\cdot\nabla\tilde{e}_s + R_{nw}\nabla\cdot\nabla\tilde{e}_w + R_{nn}\nabla\cdot\nabla\tilde{e}_n) - i\omega b_n(\tilde{e}_n - \tilde{e}_s) = -n_0^{(n)}\omega^2\gamma_0^{(n)}\tilde{e}_n \tag{49c}$$

For harmonic plane P waves propagate in the porous medium, the following solutions for the dilatations can be assumed

$$\tilde{e}_s = A_s e^{i(\omega t - k_p \mathbf{n}\cdot\mathbf{x})}, \quad \tilde{e}_w = A_w e^{i(\omega t - k_p \mathbf{n}\cdot\mathbf{x})}, \quad \tilde{e}_n = A_n e^{i(\omega t - k_p \mathbf{n}\cdot\mathbf{x})} \tag{50a,b,c}$$

where $\mathbf{n}$ is the wave vector of the P waves, $k_p$ is the wave number of the P waves. In general, $k_p$ is a complex number. The phase velocity of P waves is defined as $v_p = \omega/\mathrm{Re}(k_p)$ and the attenuation coefficient of the P wave is denoted by the imaginary part of $k_p$. Substitution of (50) into (49) yields the homogeneous linear equations for $A_s$, $A_w$, $A_n$. The condition of existence of non-trivial solutions for the homogeneous linear equations gives the dispersion relation for P waves in the porous medium. For given $\omega$, the polynomial dispersion equation derived from (49) and (50) has three roots for $k_p^2$ and thus $k_p$ has six roots. However, only three of them are physically reasonable, as in terms of (50), the imaginary part of $k_p$ should be less than zero to guarantee the decreasing amplitudes of the P waves. Consequently, for the porous medium saturated by two fluids, three P waves exist simultaneously. In this paper, the three P waves are denoted by P1, P2 and P3, the velocities of which are denoted by $v_{p1}$, $v_{p2}$, $v_{p3}$ and $v_{p1} > v_{p2} > v_{p3}$. Note that the P1 wave in this paper corresponds to the P1 wave in Biot's theory. Also, the P2 wave here corresponds to the diffusive P2 wave in Biot's theory. The P3 wave is related to the capillary pressure effect and determined by the two-fluid equations of motion.

Applying curl operator on the both sides of (48a), (48b), (48c), the following equations are obtained

$$\mu_s \nabla \cdot \nabla \widetilde{\boldsymbol{\Omega}}^{(s)} + \mathrm{i}\omega b_w(\widetilde{\boldsymbol{\Omega}}^{(w)} - \widetilde{\boldsymbol{\Omega}}^{(s)}) + \mathrm{i}\omega b_n(\widetilde{\boldsymbol{\Omega}}^{(n)} - \widetilde{\boldsymbol{\Omega}}^{(s)}) = -n_0^{(s)}\omega^2\gamma_0^{(s)}\widetilde{\boldsymbol{\Omega}}^{(s)} \tag{51a}$$

$$\mathrm{i}b_w(\widetilde{\boldsymbol{\Omega}}^{(w)} - \widetilde{\boldsymbol{\Omega}}^{(s)}) = n_0^{(w)}\omega\gamma_0^{(w)}\widetilde{\boldsymbol{\Omega}}^{(w)} \tag{51b}$$

$$\mathrm{i}b_n(\widetilde{\boldsymbol{\Omega}}^{(n)} - \widetilde{\boldsymbol{\Omega}}^{(s)}) = n_0^{(n)}\omega\gamma_0^{(n)}\widetilde{\boldsymbol{\Omega}}^{(n)} \tag{51c}$$

where $\widetilde{\boldsymbol{\Omega}}^{(s)} = \nabla \times \tilde{\mathbf{u}}^{(s)}$, $\widetilde{\boldsymbol{\Omega}}^{(w)} = \nabla \times \tilde{\mathbf{u}}^{(w)}$ and $\widetilde{\boldsymbol{\Omega}}^{(n)} = \nabla \times \tilde{\mathbf{u}}^{(n)}$. Substituting from (51b), (51c) for $\widetilde{\boldsymbol{\Omega}}^{(w)}, \widetilde{\boldsymbol{\Omega}}^{(n)}$ into (51a) gives the Helmholtz equation for $\widetilde{\boldsymbol{\Omega}}^{(s)}$, from which only one physically reasonable root for shear wave number $k_s$ can be obtained. As a result, there is only one shear wave exists in the porous medium saturated by two fluids. Similarly, the phase velocity of the shear wave is defined as $v_s = \omega/\mathrm{Re}(k_s)$ and imaginary part of $k_s$ represents the attenuation coefficient of the shear wave.

### *6.2. Numerical results of velocities and attenuation for plane harmonic waves*

In this section, the numerical results of velocities and attenuation for some examples will be presented. In principle, all the parameters involved in our model should be estimated from the measurable parameters. However, the concern of this section is the influence of some parameters on the velocities and the attenuations of the wave modes rather than experiments. Consequently, the parameters used in the examples of this section are not based on experiment results. We shall discuss the method to approximate some parameters involved in our model, which is useful for the application of our theory when experimental results are unavailable.

#### *6.2.1. Approximation of the parameters in the model*

In order to evaluate the parameter $K_{cp}$ in (23) and (25), empirical models, such as the van Genuchten model (van Genuchten, 1980) and the Brooks and Corey model (Brooks and Corey, 1966), relating the capillary pressure to the saturation of the wetting phase can be used. The van Genuchten model (van Genuchten, 1980) is defined as follows:

$$\overline{S}_w = \begin{cases} [1 + (\alpha p_{n-w})^n]^{-m}, & p_{n-w} > 0 \\ 1, & p_{n-w} \leqslant 0 \end{cases} \tag{52a}$$

$$\overline{S}_w = \frac{S_w - S_{wr}}{S_{ws} - S_{wr}} \tag{52b}$$

where $S_w$ is the wetting phase saturation, $S_{wr}$ is the irreducible, residual wetting phase saturation, $S_{ws}$ is the wetting phase saturation for $p_{n-w} = 0$, $p_{n-w}$ is the difference between the pressures of the wetting phase and the non-wetting phase and $m$, $n$, $\alpha$ are empirical parameters for the model. Note that Eq. (52) is valid for the non-linear equilibrium state. For linear case, since the infinitesimal disturbance assumption has been made, it is reasonable to assume that the disturbance always satisfies the inequality $p_{n-w} > 0$. Consequently, for infinitesimal disturbance, the relationship between the capillary pressure increment and the saturation increment can be obtained by differentiation of (52a)

$$p^{(n)} - p^{(w)} = -\frac{\eta_a}{mn\alpha(\eta_a S_w - \eta_b)^{1+1/m}(\alpha p_{n-w})^{n-1}} S \tag{53}$$

where $\eta_a = 1/(S_{ws} - S_{wr})$, $\eta_b = S_{wr}/(S_{ws} - S_{wr})$. Comparison of (53) with (25) gives the expression for the parameter $K_{cp}$

$$K_{cp} = \frac{\phi \eta_a}{mn\alpha n_0^{(w)} n_0^{(n)} (\eta_a S_w - \eta_b)^{1+1/m}(\alpha p_{n-w})^{n-1}} \tag{54}$$

The Brooks and Corey model (Brooks and Corey, 1966) is defined as follows:

$$\overline{S}_w = \begin{cases} (\varepsilon p_{n-w})^{-\lambda}, & \varepsilon p_{n-w} > 1 \\ 1, & \varepsilon p_{n-w} \leqslant 1 \end{cases} \tag{55}$$

where $\lambda$, $\varepsilon$ are empirical parameters in the model. As mentioned above, for linear case, we assume that the disturbance always satisfy the inequality $\varepsilon p_{n-w} > 1$. Similarly, the relationship between the capillary pressure increment and the saturation increment can be obtained by differentiation of (55)

$$p^{(n)} - p^{(w)} = -\frac{\eta_a}{\lambda\varepsilon(\eta_a S_w - \eta_b)^{1+1/\lambda}} S \tag{56}$$

Again, comparison of (56) with (25) gives the expression for the parameters $K_{cp}$

$$K_{cp} = \frac{\phi \eta_a}{n_0^{(w)} n_0^{(n)} \lambda\varepsilon(\eta_a S_w - \eta_b)^{1+1/\lambda}} \tag{57}$$

After determining $K_{cp}$, the capillary pressure relaxation coefficient $K_{ds}$ can be calculated by (32). Since the $K_{ds}$ is a frequency independent parameter by definition, consequently, in terms of (32), the equivalent expression for $K_{ds}$ is

$$K_{ds} = n_0^{(w)} n_0^{(n)} \xi_{\alpha_0} K_{cp} \tag{58}$$

where $\xi_{\alpha_0} = \tan(\alpha_0)/\omega$ and has a dimension of time.

The undrained modulus $K_{ud}$ defined in (36) can be approximated by the following procedure: first use Wood's formula (Wood, 1955) to calculate the average bulk modulus of the two fluids, then use the average bulk modulus of the two fluids as well as the Gassmann equation (Gassmann, 1951) to calculate the undrained bulk modulus for the two-fluid saturated porous medium. Note that when use the Gassmann equation to evaluate the undrained modulus, the bulk modulus of the mineral material making up the rock matrix is required. So when the bulk modulus of the mineral material is unknown, the undrained modulus $K_{ud}$ for the porous medium is difficult to estimate.

For the quasi-unjacketed bulk modulus $K_{uj}$ defined in (41), the bulk modulus ($K_m$) of the mineral material making up the solid phase can be used as the reference value.

For the fluid content coefficients defined in (41), the values for the case in which the couplings between the three phases are neglected can be used as reference values. In this case, the fluid content coefficients are given by

$$\beta_{\mathrm{w}} = n_0^{(\mathrm{w})}\left(\frac{1}{K_{\mathrm{w}}} - \frac{1}{K_{\mathrm{m}}}\right), \quad \beta_{\mathrm{n}} = n_0^{(\mathrm{n})}\left(\frac{1}{K_{\mathrm{n}}} - \frac{1}{K_{\mathrm{m}}}\right) \tag{59a,b}$$

where $K_{\mathrm{w}}$, $K_{\mathrm{n}}$ are the bulk moduli for the separate wetting and the non-wetting fluid. Clearly, if couplings are taken into account, the fluid content coefficients are different from (59) and only can be determined by the above-mentioned procedure.

Moreover, when evaluating the parameters $K_{\mathrm{ud}}$, $B_{\mathrm{w}}$, $B_{\mathrm{n}}$, $K_{\mathrm{uj}}$, $\beta_{\mathrm{w}}$, $\beta_{\mathrm{n}}$, Eq. (43) can be used to calculate one parameter from the other five parameters.

The relative permeability for fluid $k_{\mathrm{r}}^{(f)}$ is dependent on the saturation of the fluid. In this paper, the Brooks and Corey model (Brooks and Corey, 1966) is used to determine $k_{\mathrm{r}}^{(f)}$. For a porous medium saturated by two fluids, in terms of the Brooks and Corey model (Brooks and Corey, 1966), the relative permeabilities for the wetting and the non-wetting phase have the following expressions

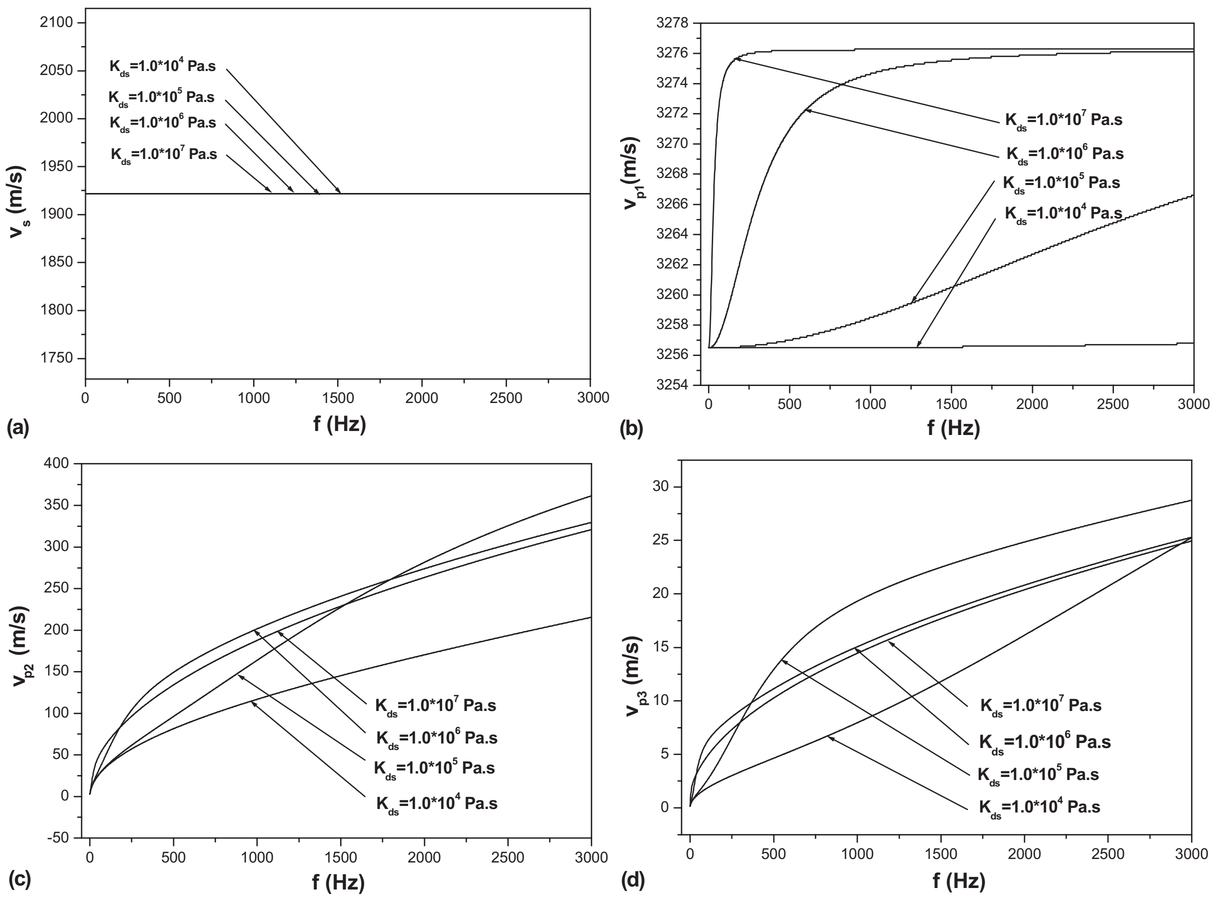


Fig. 2. Velocities of the shear wave, P1, P2 and P3 wave in the porous medium with varying capillary pressure relaxation coefficient $K_{\mathrm{ds}}$: (a) velocity for the shear wave ($v_{\mathrm{s}}$); (b) velocity for the P1 wave ($v_{\mathrm{p1}}$); (c) velocity for the P2 wave ($v_{\mathrm{p2}}$); (d) velocity for the P3 wave ($v_{\mathrm{p3}}$).

$$k_r^{(w)} = \overline{S}_w^{l+1+2/\lambda}, \quad k_r^{(n)} = (1-\overline{S}_w)^l(1-\overline{S}_w^{l+2/\lambda}) \tag{60a,b}$$

where $\overline{S}_w$ is given by (52b) and $l$ is a model parameter.

### 6.2.2. Influence of the capillary pressure relaxation coefficient on the velocities and the attenuation

In this section, the influences of the capillary pressure relaxation coefficient on the velocities and the attenuation coefficients of the four wave modes in the porous medium are considered. In order to evaluate the parameter $K_{cp}$ in (23), Eq. (57) derived from the Brooks and Corey model (Brooks and Corey, 1966) is used in this section. The following values of the parameters are assumed $\lambda = 1.5$, $\varepsilon = 5.0667\times10^{-9}\,\mathrm{Pa}^{-1}$, and $S_{wr} = 0.1$, $S_{ws} = 0.9$ respectively. The parameters for the solid skeleton are as follows: the density $\gamma_0^{(s)} = 2.339\times10^3\,\mathrm{kg/m^3}$, the porosity $\phi = 0.3$, the elastic coefficients $K_s = 10.0\,\mathrm{GPa}$, $\mu_s = 7.0\,\mathrm{GPa}$, the permeability $k = 1.0\times10^{-12}\,\mathrm{m^2}$. Note that the elastic coefficients $K_s$, $\mu_s$ relate to the drained bulk and shear modulus of the porous sample by (34) and (33). The parameters for the wetting phase are assumed the following values: $\gamma_0^{(w)} = 1.0\times10^3\,\mathrm{kg/m^3}$, the viscosity $\eta^{(w)} = 1.0\times10^{-3}\,\mathrm{Pa\,s}$ and the relative permeability $k_r^{(w)}$ is given by (60a) with $l = 1$. For the non-wetting phase, the parameters are as follows:

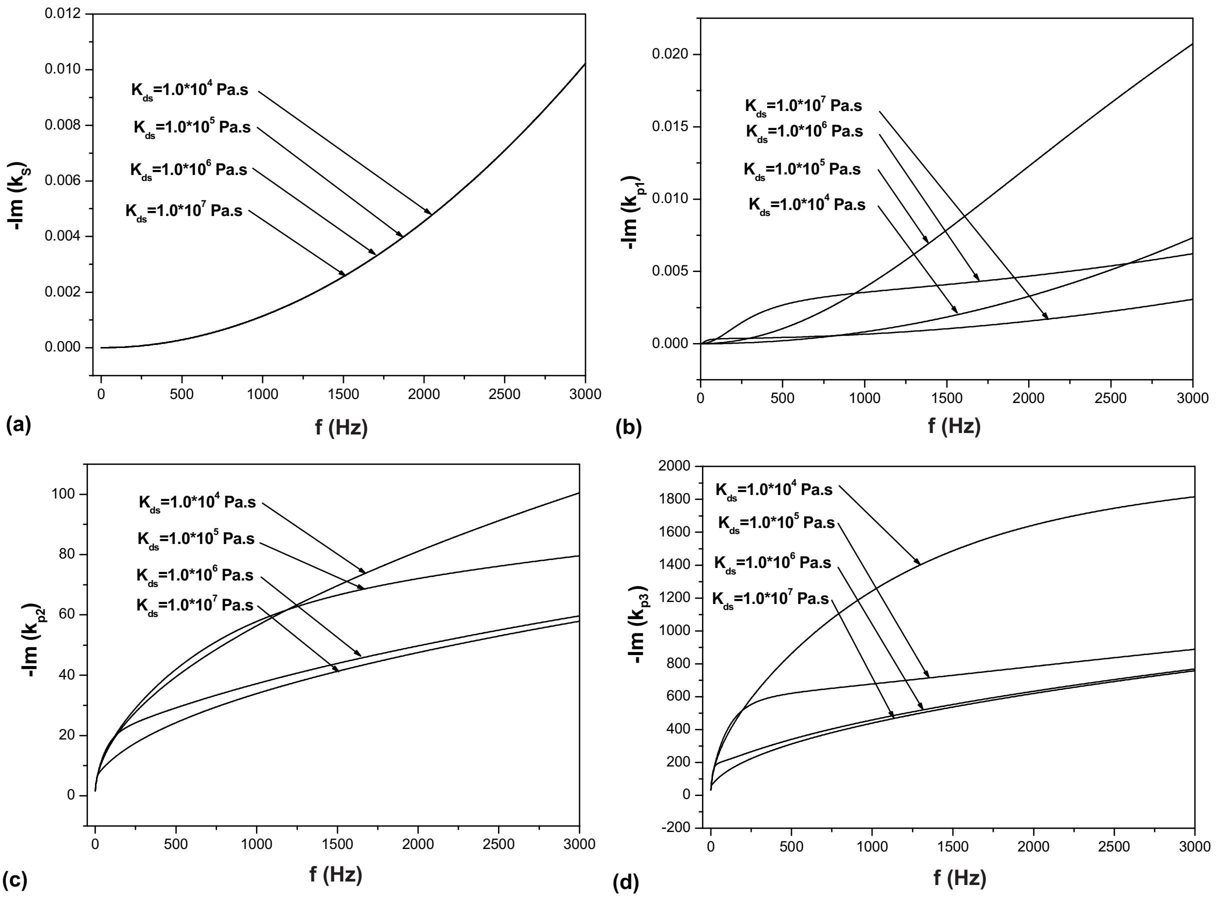


Fig. 3. Attenuation coefficients of the shear wave, P1, P2 and P3 wave in the porous medium with varying capillary pressure relaxation coefficient $K_{ds}$: (a) attenuation coefficient for the shear wave; (b) attenuation coefficient for the P1 wave; (c) attenuation coefficient for the P2 wave; (d) attenuation coefficient for the P3 wave.

$\gamma_0^{(n)} = 0.65 \times 10^3\ \mathrm{kg/m^3}$, the viscosity $\eta^{(n)} = 1.0 \times 10^{-1}\,\mathrm{Pa\,s}$ and the relative permeability $k_r^{(n)}$ is given by (60b). The initial saturation of the wetting phase assumed $S_0 = 0.6$. In terms of (57) and using the values of $\lambda$, $\varepsilon$ given above, one obtains $K_{cp} = 5.0\,\mathrm{GPa}$. In this example, the parameter $K_{ds}$ is equal to $1.0 \times 10^4\,\mathrm{Pa\,s}$, $1.0 \times 10^5\,\mathrm{Pa\,s}$, $1.0 \times 10^6\,\mathrm{Pa\,s}$, $1.0 \times 10^7\,\mathrm{Pa\,s}$, respectively. In calculation, we take $M_{ww} = 2.5\,\mathrm{GPa}$, $M_{sw} = 1.0\,\mathrm{GPa}$, $M_{nn} = 0.8\,\mathrm{GPa}$, $M_{sn} = 0.4\,\mathrm{GPa}$, $M_{wn} = 0.2\,\mathrm{GPa}$. In terms of (39), (40), (42) and (43), the six parameters defined in (36) and (41) for the undrained and the quasi-unjacketed test are obtained: $K_{ud} = 10.8\,\mathrm{GPa}$, $B_w = 0.19$, $B_n = 0.18$, $K_{uj} = 17.3\,\mathrm{GPa}$, $\beta_w = 5.5 \times 10^{-11}\,\mathrm{Pa^{-1}}$, $\beta_n = 1.4 \times 10^{-10}\,\mathrm{Pa^{-1}}$.

The velocities and the attenuations of the four wave modes are shown in Figs. 2 and 3. It follows from Fig. 2(a) and Fig. 3(a), the variation of the capillary pressure relaxation coefficient $K_{ds}$ has no influence on the velocity and the attenuation of the shear wave in the porous medium. Fig. 2(b) shows that with increasing $K_{ds}$, $v_{p1}$ increases slightly. It follows from Fig. 2(c) and (d), the variations of $v_{p2}$, $v_{p3}$ with increasing $K_{ds}$ are more significant than that of $v_{p1}$. Specifically, $v_{p2}$, $v_{p3}$ increase with increasing $K_{ds}$ at first, however, when $K_{ds} > 1.0 \times 10^5\,\mathrm{Pa\,s}$, $v_{p2}$, $v_{p3}$ decrease with increasing $K_{ds}$. Fig. 3(b) shows $-\mathrm{Im}(k_{p1})$ increases significantly with increasing $K_{ds}$ at first, nevertheless, when $K_{ds} > 1.0 \times 10^5\,\mathrm{Pa\,s}$, $-\mathrm{Im}(k_{p1})$ decreases quickly with increasing $K_{ds}$. Fig. 3(c) and (d) demonstrate that $-\mathrm{Im}(k_{p2})$ and $-\mathrm{Im}(k_{p3})$ decrease significantly with increasing $K_{ds}$. It follows from the calculation of this section the capillary pressure relaxation process

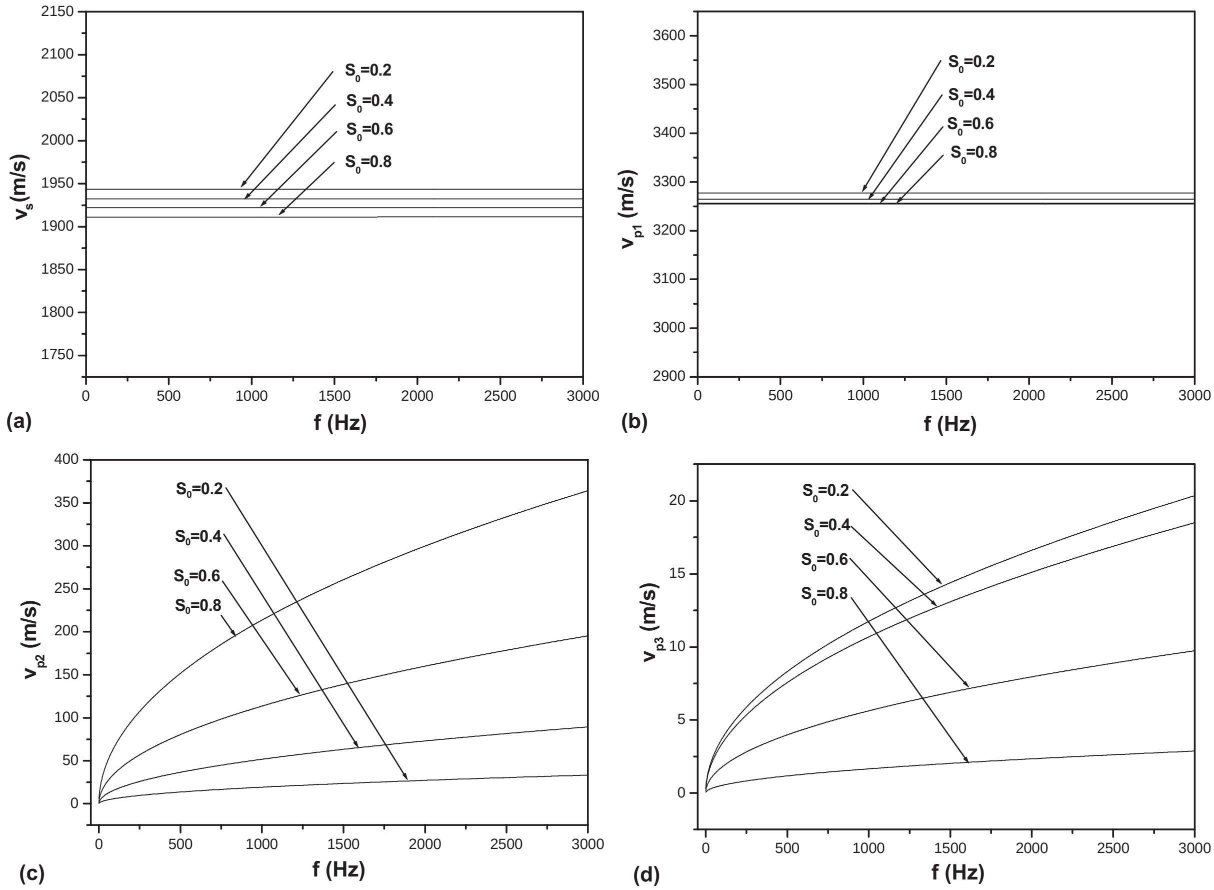


Fig. 4. Velocities of the shear wave, P1, P2 and P3 wave in the porous medium with varying saturation of the wetting phase $S_0$: (a) velocity for shear wave ($v_s$); (b) velocity for the P1 wave ($v_{p1}$); (c) velocity for the P2 wave ($v_{p2}$); (d) velocity for the P3 wave ($v_{p3}$).

has a significant influence on the attenuation of the various P wave modes. So it is another important mechanism accounting for the attenuation of the two-fluid saturated porous medium.

### 6.2.3. Influence of saturation on the velocities and the attenuation

In this section, the influence of saturation on the velocities and attenuation coefficients of the four waves in the porous medium is considered. The capillary pressure relaxation effect is neglected here, which is tantamount to the assumption that the capillary pressure equilibrium is reached instantaneously. We assume the parameters of the solid, the wetting and non-wetting phase are the same as in Section 6.2.2. In principle, the parameters $M_{ww}$, $M_{nn}$, $M_{sw}$, $M_{sn}$, $M_{wn}$ depend on the saturation of the wetting phase. However, for the simplicity of the analysis, in this section, in view of (19), we assume the parameters $M_{ww}$, $M_{nn}$, $M_{sw}$, $M_{sn}$, $M_{wn}$ are independent of the saturation. As a result, as in Section 6.2.2, the five parameters take the following values: $M_{ww} = 2.5\,\text{GPa}$, $M_{sw} = 1.0\,\text{GPa}$, $M_{nn} = 0.8\,\text{GPa}$, $M_{sn} = 0.4\,\text{GPa}$, $M_{wn} = 0.2\,\text{GPa}$. The saturation values $S_0 = 0.2,\ 0.4,\ 0.6,\ 0.8$ are assumed. The parameter $K_{cp}$ is calculated by (57) and all the related parameters are assumed the same values as in Section 6.2.2. Then, in terms of (57), the values of $K_{cp}$ corresponding to the saturation $S_0 = 0.2,\ 0.4,\ 0.6,\ 0.8$ are $K_{cp} = 109.65,\ 11.70,\ 5.00,\ 4.28\,\text{GPa}$,

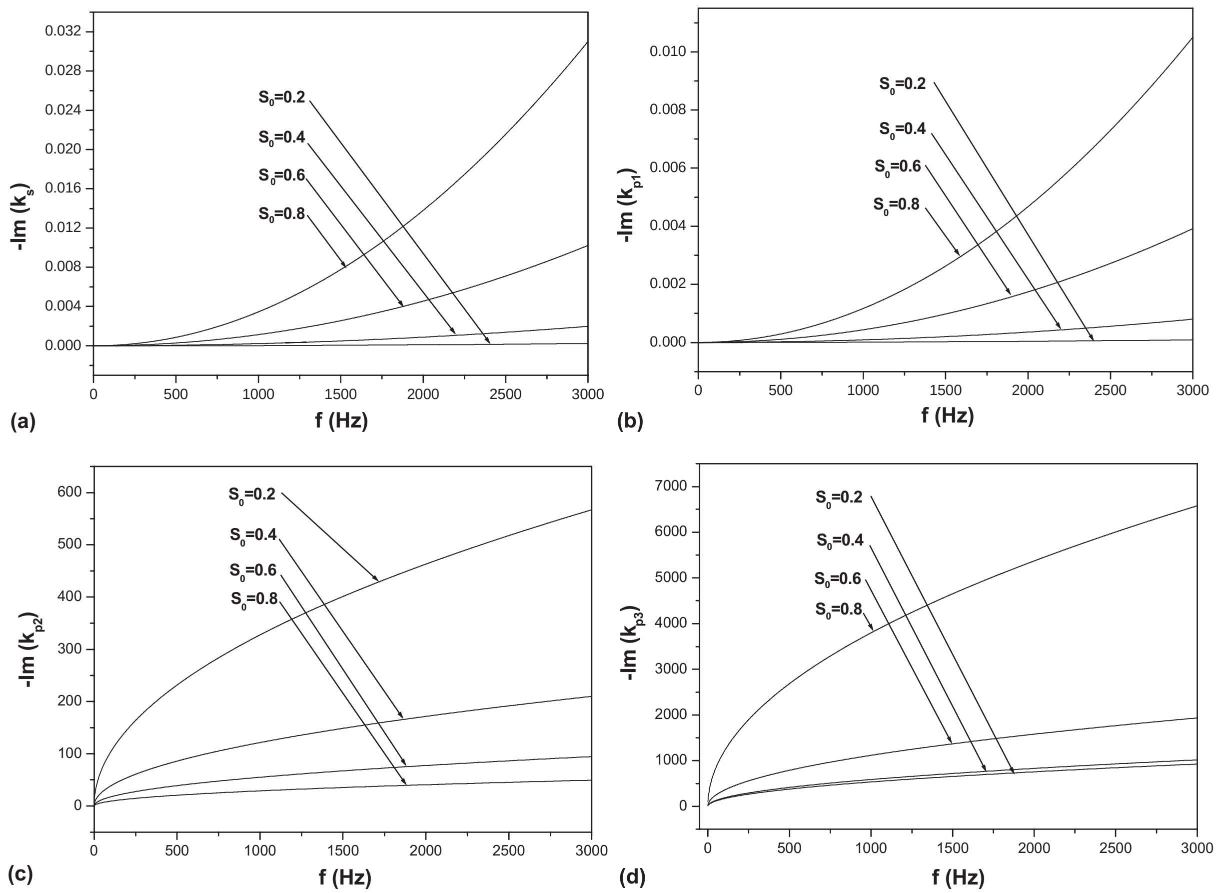


Fig. 5. Attenuation coefficients of the shear wave, P1, P2 and P3 wave in the porous medium with varying saturation of the wetting phase $S_0$: (a) attenuation coefficient for the shear wave; (b) attenuation coefficient for the P1 wave; (c) attenuation coefficient for the P2 wave; (d) attenuation coefficient for the P3 wave.

respectively. The six parameters defined in (36) and (41) for the undrained and the quasi-unjacketed test can be calculated with given $M_{ww}$, $M_{nn}$, $M_{sw}$, $M_{sn}$, $M_{wn}$, $K_{cp}$.

The velocities and the attenuations of the four wave modes in the porous medium are plotted in Figs. 4 and 5. Fig. 4(a) and (b) demonstrate that the velocities of shear wave and P1 wave only decrease slightly with increasing saturation. The velocity of P2 wave increases with increasing saturation, while the velocity of P3 wave decreases with increasing saturation. Fig. 5 shows that the attenuation coefficients of the shear wave, P1 and P3 wave increase with increasing saturation, while the attenuation coefficients of the P2 wave decreases with increasing saturation.

## 7. Conclusions

A linear dynamic model for the porous medium saturated by two fluids in the isothermal case has been developed in the paper. As expected, one kind of shear wave and three kinds of P waves are predicted by our theory. Our model can be used in the dynamic analysis of rock saturated by oil and water, rock saturated by oil and gas or soil saturated by water and gas. The suggested model is derived in a systematic way by using the entropy inequality and the balance equations from the mixture theory. Moreover, the proposed model overcomes the drawbacks of Biot's theory: it avoids the Lagrangian formulation by using balance equations of the mixture theory and thermodynamics approach. It eliminates the phase separation assumption by introducing a single free energy constitutive function for the porous medium. In other words, our theory combines the advantage of one energy concept in Biot's theory with the volume fraction concept and balance equations of the mixture theory. For the application of our theory, some new experiments have been introduced to evaluate the parameters involved in our model. In terms of the introduced experiments, the approaches have been devised to determine the model parameters. It has been shown that the capillary pressure relaxation coefficient has significant influence on the attenuation of the P waves but it has no influence on the speed and the attenuation of the S wave. As a result, unlike Biot's one-fluid model, our model includes two mechanisms accounting for the attenuation of the two-fluid saturated porous medium: the common drag force mechanism and the present capillary pressure relaxation mechanism. Also, the numerical results of the paper show the saturation of the wetting phase has influence on the speeds and attenuations of both the S wave and the P waves.

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